Meadows: How to divide by zero

João Dias[1], Bruno Dinis[1]

[1] *Department of Mathematics and CIMA, University of Évora, Portugal*

Email: joao.miguel.dias@uevora.pt; bruno.dinis@uevora.pt

Abstract:

Meadows were introduced by Bergstra and Tucker in [5] as algebraic structures, given by equational theories, where it makes sense to divide by zero. To be more specific, a meadow is a sort of commutative ring with a multiplicative identity element and a total multiplicative inverse operation. Two of the main classes of meadows are involutive meadows, in which the inverse of zero is zero, and common meadows (introduced in [3]), in which the inverse of zero is a term **a** which is maximal, in the sense that $x + \mathbf{a} = \mathbf{a}$, for every element x in the meadow. In common meadows $0 \cdot x$ doesn't have to be equal to 0.

Even though meadows were only recently introduced, the subject is revealing to be of interest, mostly as datatypes given by equational axiomatizations (see e.g. [2, 5, 4, 1]) allowing for simple term rewriting systems which are easier to automate in formal reasoning [1, 6]. More recently, E. Bottazzi and B. Dinis [7] found a connection with nonstandard analysis which provides new models for both involutive and common meadows. Also, J. Dias and B. Dinis in [8] and [9] studied the algebraic properties of common meadows and obtained some progress towards an enumeration of finite meadows.

Since this is a emergent field in mathematics there are many open questions, and possible lines of research. For example, study common meadows not only from an algebraic point of view but also from a topological point of view, via topological meadows; study related structures were some axioms are relaxed (e.g. commutativity or distributivity); the study of combinatorial/number theoretical problems related with meadows; the study of the model theory of meadows and of its complexity.

References

[1] Jan Bergstra. Arithmetical datatypes, fracterms, and the fraction definition problem. *Transmathematica*, 2020.

[2] Jan Bergstra, Yoram Hirshfeld, and John Tucker. *Fields, Meadows and Abstract Data Types*, pages 166–178. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

[3] Jan Bergstra and Alban Ponse. *Division by Zero in Common Meadows*, pages 46–61. Springer International Publishing, Cham, 2015.

[4] Jan Bergstra and Alban Ponse. Arithmetical datatypes with true fractions. *Acta Inform.*, 57(3-5):385–402, 2020.

[5] Jan Bergstra and John Tucker. The rational numbers as an abstract data type. *J. ACM*, 54(2):7–es, apr 2007.

[6] Jan Bergstra and John Tucker. On the axioms of common meadows: Fracterm calculus, flattening and incompleteness. *The Computer Journal*, 66(7):1565–1572, 2023.

[7] Emanuele Bottazzi and Bruno Dinis. Flexible involutive meadows, 2023. (submitted).

[8] João Dias and Bruno Dinis. Strolling through common meadows. *arXiv preprint arXiv:2311.05460*, 2023.

[9] João Dias and Bruno Dinis. Enumerating finite common meadows. *arXiv preprint arXiv:2401.10631*, 2024.